VANISHING CYCLES AND MUTATION

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ABSTRACT. Using Floer cohomology, we establish a connection between Picard-Lefschetz theory and the notion of mutation of exceptional collections in homological algebra.

1. Introduction

This talk is about symplectic aspects of Picard-Lefschetz theory, and the role of Floer cohomology in that context. I have relied on two main sources for inspiration, which are the ideas of Donaldson on vanishing cycles [3] respectively those of Kontsevich, partly in collaboration with Barannikov, on mirror symmetry for Fano varieties [8]. On a more technical level, the basis is provided by Fukaya's work on Floer cohomology [5]. I have tried to be as concrete as possible: not only are all objects mentioned rigorously defined, but they can be explicitly computed, and the assertions made about them checked, in many examples. This hands-on approach has its drawbacks, one of which will be mentioned after the summary of contents.

The basic geometric notion is that of an exact Morse fibration, a symplectic analogue of a holomorphic Morse function. One way of analyzing such fibrations is through the vanishing cycles in a fibre. When the base is a disc, one conveniently makes a choice of a finite family of vanishing cycles, forming a so-called distinguished basis. Any two such bases are connected by a sequence of Hurwitz moves. These are familiar concepts except that here vanishing cycles are considered as Lagrangian submanifolds, rather than only as homology classes.

Apart from their role as geometric objects worthy of study on their own, exact Morse fibrations are also relevant to Floer theory since, when equipped with suitable Lagrangian boundary conditions, they provide homomorphisms between Floer cohomology groups. As an application of these new maps we construct a long exact sequence analogous to that of Floer in gauge theory.

After that we return to exact Morse fibrations over a disc. The goal is to associate to each such fibration a triangulated category. An additional assumption is necessary: for simplicity, let's say that the total space of the fibration has zero first Chern class and zero first Betti number. The construction proceeds in several steps: first one chooses a distinguished basis of vanishing cycles. From that one obtains a Fukaya-type A_{∞} -category,

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unique up to quasi-isomorphism; our invariant is the derived category of this. It resembles derived categories of coherent sheaves on some Fano varieties, in that it is generated by an exceptional collection. The main point is to show that Hurwitz moves of the distinguished basis correspond to mutations of the exceptional collection, which leave the derived category unchanged. The explicit nature of mutations also allows them to be used for concrete computations of Floer cohomology.

We have to admit that our triangulated categories are only well-defined in a weak sense: different choices made during the construction lead to equivalent categories, but it has not been proved that the equivalences are canonical up to isomorphism (to have a completely satisfactory theory, it would further be necessary to establish coherence relations between functor isomorphisms). Rather than trying to do these improvements, it seems better to look for an alternative definition bypassing the choice of distinguished basis. The approach envisaged by Kontsevich is of this kind, but more work would be needed to put it on a rigorous footing.

Acknowledgments. Thanks go to Ivan Smith for reading the manuscript and suggesting several improvements.

2. Picard-Lefschetz Theory

(2A) Let (M, ω, θ) be an exact symplectic manifold of dimension 2n. This means that M is compact with boundary, $\omega \in \Omega^2(M)$ is a symplectic form, and $\theta \in \Omega^1(M)$ satisfies $d\theta = \omega$. We will consider only symplectic automorphisms ϕ of M which are equal to the identity near ∂M . Such a ϕ is called exact if $[\phi^*\theta - \theta] \in H^1(M, \partial M; \mathbb{R})$ is zero. The exact symplectic automorphisms form a subgroup $\operatorname{Symp}^e(M) \subset \operatorname{Symp}(M)$. Note that any isotopy within this subgroup is Hamiltonian.

Let S be a smooth manifold with boundary. An exact symplectic fibration over S consists of data (E, π, Ω, Θ) as follows. $\pi: E \to S$ is a proper differentiable fibre bundle whose fibres are 2n-dimensional manifolds with boundary. This means that E itself is a manifold with codimension two corners, with $\partial E = \partial_v E \cup \partial_h E$ consisting of two faces: $\partial_v E = \pi^{-1}(\partial S)$, while $\pi | \partial_h E : \partial_h E \to S$ is again a differentiable fibre bundle. $\Omega \in \Omega^2(E)$ is closed, its vertical part $\Omega | \ker(D\pi)$ nondegenerate at every point, and $\Theta \in \Omega^1(E)$ satisfies $d\Theta = \Omega$. We impose a final condition of triviality near $\partial_h E$. This means that there should be a neighbourhood $W \subset E$ of $\partial_h E$ and a diffeomorphism, for some $z \in S, \Xi : S \times (W \cap E_z) \to W$ lying over S, such that $\Xi^*\Omega = pr_2^*(\Omega|E_z)$ and $\Xi^*\Theta = pr_2^*(\Theta|E_z)$; here pr_2 is projection from $S \times (W \cap E_z)$ to the second factor. Clearly each fibre $(E_z, \omega_z = \Omega | E_z, \theta_z = \Theta | E_z)$ is an exact symplectic manifold. The form Ω defines a canonical connection on $\pi: E \to S$, with structure group $\operatorname{Symp}^e(E_z)$. In fact there is a bijective correspondence between fibre bundles with structure group $\operatorname{Symp}^e(E_z)$ in the usual sense, and cobordism classes of exact

symplectic fibrations. We denote the parallel transport maps of the canonical connection by $\rho_c: E_{c(a)} \to E_{c(b)}$, for $c: [a; b] \to S$.

(2B) From now on assume that S is two-dimensional and oriented. An **exact Morse fibration** (this is shorthand for "exact symplectic fibration with Morse-type critical points") over S consists of data $(E, \pi, \Omega, \Theta, J_0, j_0)$. The properties of E, π, Ω, Θ are as before, except that π is allowed to have finitely many critical points. Each fibre may contain at most one of these points, and there should be none at all on ∂E . J_0 is an integrable complex structure defined in a neighbourhood of the set $E^{\text{crit}} \subset E$ of critical points, and Ω must be a Kähler form for it. Similarly j_0 is a positively oriented complex structure on a neighbourhood of the set $S^{\text{crit}} \subset S$ of critical values. They should be such that π is (J_0, j_0) -holomorphic, with nondegenerate second derivative at each critical point. We will usually denote exact Morse fibrations by (E, π) only.

One thing that needs explaining is why these are supposed to be analogues of holomorphic functions. For this one considers pairs (j, J) consisting of a positively oriented complex structure j on S and an almost complex structure J on E, such that $j=j_0$ near $S^{\rm crit}$, $J=J_0$ near $E^{\rm crit}$, π is (J,j)-holomorphic, and $\Omega(\cdot,J\cdot)|\ker(D\pi)$ is symmetric and positive definite everywhere. In this situation we say that J is compatible relative to j. The space of such pairs (j,J) is always contractible, and in particular nonempty. Moreover, for a fixed pair, by adding a positive two-form from S one can modify Ω such that it becomes symplectic and tames J.

Restricting any exact Morse fibration to $S \setminus S^{\text{crit}}$ yields an exact symplectic fibration. Before bringing the singular fibres into the picture, we need some more definitions. Let (M, ω, θ) be an exact symplectic manifold. A Lagrangian submanifold $L \subset M$, always assumed to be disjoint from ∂M , is called exact if $[\theta|L] \in H^1(L;\mathbb{R})$ is zero. A **framed Lagrangian sphere** is a Lagrangian submanifold L together with an equivalence class [f] of diffeomorphisms $f: S^n \to L$. Here f_1, f_2 are equivalent if $f_2^{-1}f_1$ is isotopic to some element of $O(n+1) \subset \text{Diff}(S^n)$. One can associate to any (L, [f]) a Dehn twist $\tau_{(L,[f])} \in \text{Symp}(M)$ which is unique up to Hamiltonian isotopy. If L is exact, so is the Dehn twist along it. In future, we will often omit the framing [f] from the notation.

To return to our discussion, let (E, π) be an exact Morse fibration. Take a path $c: [0;1] \to S$ with $c^{-1}(S^{\text{crit}}) = \{1\}$ and $c'(1) \neq 0$. Let x be the unique critical point in $E_{c(1)}$. Then the stable manifold

$$B = \{ y \in E_{c(s)}, \ 0 \le s < 1, \text{ with } \lim_{t \to 1} \rho_{c|[s,t]}(y) = x \} \cup \{x\}$$

is a smoothly embedded (n+1)-dimensional ball on which Ω vanishes identically, and therefore $V = \partial B = B \cap E_{c(0)}$ is an exact Lagrangian submanifold of $E_{c(0)}$ diffeomorphic to S^n . Moreover, V has a canonical structure of a framed Lagrangian sphere, constructed by first carrying V by parallel transport to $B \cap E_{c(s)}$, for some s close to 1, then projecting orthogonally in local

Kähler coordinates to TB_x , and finally projecting radially to the unit sphere in that tangent space. The composition of these maps is a diffeomorphism $f^{-1}: V \to S^n$ whose inverse is the framing. (V, [f]) is called the **vanishing cycle** associated to c. A symplectic version of the Picard-Lefschetz theorem says that for l, c as in Figure 1, the monodromy $\rho_l \in \operatorname{Symp}^e(E_{c(0)})$ is isotopic to $\tau_{(V, [f])}$.

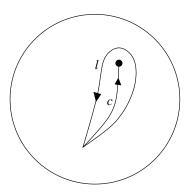


FIGURE 1.

(2c) Suppose now that S = D is the closed unit disc in \mathbb{C} . Let (E, π) be an exact Morse fibration over it with m critical values. Let M be the fibre at some base point $z_0 \in \partial D$, say $z_0 = -i$. An admissible choice of paths is a family (c_1, \ldots, c_m) looking as in Figure 2, ordered by their tangent directions at z_0 . The collection of exact framed Lagrangian spheres in M arising from them is called a **distinguished basis** of vanishing cycles. Modifying the paths affects the distinguished basis in a way which can be determined using the Picard-Lefschetz theorem. The outcome is encoded in the following abstract notion:

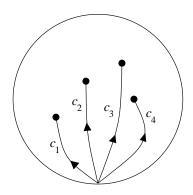


Figure 2.

Definition 2.1. A Lagrangian configuration in an exact symplectic manifold M is an ordered family $\Gamma = (L_1, \ldots, L_m)$ of exact framed Lagrangian spheres. Two configurations are Hurwitz equivalent if they can be connected by a sequence of the following moves and their inverses:

- $\Gamma = (L_1, \ldots, L_m) \rightsquigarrow (L_1, \ldots, L_{i-1}, \phi(L_i), L_{i+1}, \ldots, L_m)$ for some $1 \le i \le m$ and some $\phi \in \operatorname{Symp}^e(M)$ isotopic to the identity;
- $\Gamma \leadsto c\Gamma = (\tau_{L_1}(L_2), \tau_{L_1}(L_3), \dots, \tau_{L_1}(L_m), L_1);$
- $\Gamma \leadsto r\Gamma = (L_1, \dots, L_{m-2}, \tau_{L_{m-1}}(L_m), L_{m-1}).$

Thus, the Hurwitz equivalence class of a distinguished basis is an invariant of the exact Morse fibration (E, π) . Conversely, from M and that Hurwitz equivalence class one can reconstruct the fibration up to a suitable notion of deformation equivalence.

References. The symplectic nature of Dehn twists in all dimensions was first noticed by Arnol'd [1]. Global properties of these maps are discussed in [14], [15], [16]. I am not aware of any systematic exposition of symplectic Picard-Lefschetz theory in the literature.

3. Floer cohomology

(3A) From now on, any exact symplectic manifold (M, ω, θ) is assumed to have contact type boundary, with $\theta | \partial M$ being the contact one-form (the same condition will be imposed on $\Theta | \partial E_z$, for E_z any fibre of a Morse fibration). Then for any pair (L_1, L_2) of exact Lagrangian submanifolds in M there is a well-defined Floer cohomology group $HF(L_1, L_2)$, which is a finite-dimensional vector space over the field $\mathbb{Z}/2$. We remind the reader that this is invariant under isotopies of L_1 or L_2 , satisfies $HF(\phi L_1, \phi L_2) \cong HF(L_1, L_2)$ for any $\phi \in \operatorname{Symp}^e(M)$, and that there is a natural Poincaré duality $HF(L_1, L_2) \cong HF(L_2, L_1)^{\vee}$.

As a warm-up exercise, suppose that we have a compact oriented surface S with boundary, and an exact Morse fibration (E^{2n+2},π) over it. A Lagrangian boundary condition for E is a closed submanifold $Q^{n+1} \subset \partial_v E \setminus \partial_h E$ such that $\pi|Q:Q\to\partial S$ is a smooth fibration, satisfying $\Omega|Q=0$ and $[\Theta|Q]=0\in H^1(Q;\mathbb{R})$. Then the intersection $Q_z=Q\cap E_z$, for any $z\in\partial S$, is an exact Lagrangian submanifold in E_z , and parallel transport along ∂S takes these Lagrangian submanifolds into each other. Choose a complex structure j on S and an almost complex structure J on E which is compatible relative to j, as defined in the previous section. There is a Gromov type invariant $\Phi(E,\pi,Q)\in\mathbb{Z}/2$ which counts, in the familiar sense, the number of (j,J)-holomorphic sections $u:S\to E$ with $u(\partial S)\subset Q$. The exactness assumptions imply that there can be no bubbles (J-holomorphic spheres in a fibre E_z , or J-holomorphic discs in E_z with boundary in Q_z), so that the definition of the invariant is technically quite simple.

Example 3.1. Let L be an exact framed Lagrangian sphere in an exact symplectic manifold M. Starting from a standard local model, one can construct

an exact Morse fibration (E,π) over D with $E_{z_0} = M$ for $z_0 = -i \in \partial D$, having exactly one critical point, such that the monodromy around ∂D is τ_L . Because $\tau_L(L) = L$, there is a unique Lagrangian boundary condition $Q \subset E$ with $Q_{z_0} = L$. $\Phi(E,\pi,Q)$ vanishes because the expected dimension of the space of (j,J)-holomorphic sections is always odd, hence never zero.

Now let S be as before but with a finite set of marked points $\Sigma \subset \partial S$. Suppose moreover that around each $\zeta \in \Sigma$ we have preferred local coordinates, given by an oriented embedding $\psi_{\zeta}: D^+ \to S$ of the halfdisc $D^+ = D \cap \{ \operatorname{im}(z) \geq 0 \}$ with $\psi_{\zeta}(0) = \zeta$. Let (E, π) be an exact Morse fibration over $S^* = S \setminus \Sigma$ which is trivial near the marked points. This means that we have a fixed exact symplectic manifold Mand preferred embeddings $\Psi_{\zeta}: (D^+ \setminus \{0\}) \times M \to E$ lying over ψ_{ζ} , satisfying some obvious conditions concerning Ω,Θ that we do not care to write down. If $Q \subset E$ is a Lagrangian boundary condition, there is for each $\zeta \in \Sigma$ a unique pair $L_{\zeta,\pm}$ of exact Lagrangian submanifolds of M such that $\Psi_{\zeta}^{-1}(Q) = [-1;0) \times L_{\zeta,-} \cup (0;1] \times L_{\zeta,+}$. After choosing a complex structure j on S^* such that the ψ_{ζ} become holomorphic, and an almost complex structure J on E which is compatible relative to j and satisfies some additional conditions with regard to Ψ_{ζ} , one can count pseudo-holomorphic sections with suitable behaviour near the marked points. The outcome is a relative invariant

(1)
$$\Phi_{rel}(E, \pi, Q) \in \bigotimes_{\zeta \in \Sigma} HF(L_{\zeta,+}, L_{\zeta,-}).$$

These invariants satisfy the standard gluing law for a topological quantum field theory, which one can formulate in two parts as follows. First, if S is not connected then the relative invariant decomposes into the tensor product of relative invariants associated to its connected components. Second, suppose that there are two marked points $\zeta, \zeta' \in \Sigma$ with

$$L_{\zeta,\pm} = L_{\zeta',\mp}.$$

One can define a new surface \overline{S} by removing small half-discs around ζ,ζ' and gluing together the resulting half-circles, as in Figure 3. There is a natural set of marked points $\overline{\Sigma} \subset \partial \overline{S}$ which is inherited from $\Sigma \setminus \{\zeta,\zeta'\}$. A similar process applied to (E,π) constructs a new exact Morse fibration over $\overline{S} \setminus \overline{\Sigma}$ with Lagrangian boundary conditions. The gluing rule says that on the level of the invariants Φ_{rel} this translates into contracting $HF(L_{\zeta,+},L_{\zeta,-}) \otimes HF(L_{\zeta',+},L_{\zeta',-})$ by Poincaré duality.

Remark 3.2. The reader is hereby warned of two possible misunderstandings: the gluing process does not take place along the boundaries ∂E_z of the fibres, and neither do we glue together two boundary circles of S.

(3B) We next give three examples. Throughout, Poincaré duality will be used freely to write the invariants Φ_{rel} in various equivalent ways, e.g. as multilinear maps between Floer cohomology groups.

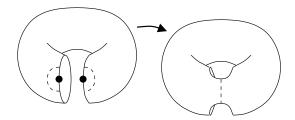


Figure 3.

- Let M be an exact symplectic manifold and $L_1, L_2, L_3 \subset M$ exact Lagrangian submanifolds. Take S = D and $\Sigma = \{three\ points\} \subset \partial S$. Let I_1, I_2, I_3 be the three components of ∂S^* , ordered in positive sense. The trivial fibration $E = S^* \times M$ with Lagrangian boundary conditions $Q = \bigcup_{\nu=1}^3 I_{\nu} \times L_{\nu}$ yields a relative invariant which can be written as a map $HF(L_2, L_3) \otimes HF(L_1, L_2) \to HF(L_1, L_3)$. This is just a reformulation of the usual "pair-of-pants" product. Reversing the orientation of S yields another relative invariant, which is the Poincaré dual coproduct $HF(L_1, L_3) \to HF(L_2, L_3) \otimes HF(L_1, L_2)$.
- Let M be an exact symplectic manifold, $L_1, L_2 \subset M$ exact Lagrangian submanifolds, and $\phi \in \operatorname{Symp}^e(M)$ an automorphism which is isotopic to the identity. Imagine S = D as being constructed out of a smaller disc D_0 and two other pieces $D_1, D_2 = [0;1]^2$. The marked points are $\Sigma = \{\zeta_1, \zeta_2\}$, where ζ_1 is obtained by identifying $(0,1) \in D_1$ with $(0,0) \in D_2$, and ζ_2 is similarly $(1,1) \in D_1$ or $(1,0) \in D_2$. One can construct an exact symplectic fibration E_0 over D_0 , which is topologically $D_0 \times M$ but has nontrivial forms Ω and Θ , such that the monodromy around ∂D_0 is ϕ . Assemble E_0 and the two trivial fibrations $E_1 = (D_1 \setminus \{(0,1),(1,1)\}) \times M$, $E_2 = (D_2 \setminus \{(0,0),(0,1)\}) \times M$ following the instructions in Figure 4. This gives an exact Morse fibration over S^* ; we equip it with the Lagrangian boundary condition which is the union of $(0;1) \times \{1\} \times L_1 \subset E_1$ and $(0;1) \times \{0\} \times L_2 \subset E_2$. The resulting relative invariant is a map $HF(L_1, L_2) \to HF(L_1, \phi(L_2))$, in fact just the familiar "continuation" isomorphism.
- Let M, L_1, L_2 be as before, and $L \subset M$ an exact framed Lagrangian sphere. The central piece E_0 in the previous construction can be replaced by the Morse fibration from Example 3.1. This leads to a map $HF(L_1, L_2) \to HF(L_1, \tau_L(L_2))$, which is not an isomorphism in general. In fact there seems to be no way at all of getting from our formalism a map in the inverse direction; the reason is the presence of critical points, which prevents one from reversing the orientation of the base.

Theorem 3.3. Let (M, ω, θ) be an exact symplectic manifold, $L_1, L_2 \subset M$ exact Lagrangian submanifolds, and $L \subset M$ an exact framed Lagrangian

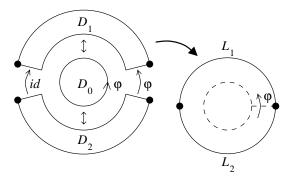


Figure 4.

sphere. Suppose that $2c_1(M, L) \in H^2(M, L)$ is zero. Then there is a long exact sequence, with a the pair-of-pants product and b the map defined in the last example above,

(2)
$$HF(L, L_2) \otimes HF(L_1, L) \xrightarrow{a} HF(L_1, L_2) \downarrow b \\ HF(L_1, \tau_L(L_2)).$$

To understand how this works, one needs to look at the Floer cochain complexes CF. To simplify, we suppress the dependence of these complexes on various additional choices, and write a,b for the maps between them inducing the Floer cohomology maps mentioned above. The central object in the proof is a map of complexes

(3)
$$\operatorname{Cone}(a) \xrightarrow{(h,b)} CF(L_1, \tau_L(L_2)).$$

Here $h: CF(L, L_2) \otimes CF(L_1, L) \to CF(L_1, \tau_L(L_2))$ is a chain homotopy $b \circ a \simeq 0$ defined as follows. Consider the exact Morse fibration with S = D, $\Sigma = \{three\ points\}$, which is represented schematically, together with its Lagrangian boundary condition, in Figure 5. Moving the two leftmost marked points simultaneously along ∂D , in a way that preserves the symmetry of the picture with respect to the x-axis, yields a one-parameter family of fibrations, and h arises from the corresponding parametrized spaces of pseudo-holomorphic sections. Once (3) has been defined, an argument using the natural filtration of the Floer complexes by the action functional shows that it is a quasi-isomorphism; the long exact sequence is an immediate consequence.

As one can see from this sketch of the argument, the definition of the third arrow in (2) uses the inverse of the quasi-isomorphism (3). Poincaré duality yields a symmetry of the exact sequence, which suggests a more

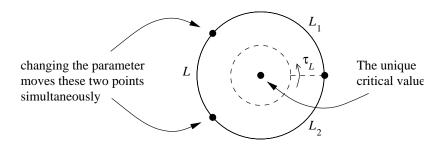


Figure 5.

direct description of that arrow. Namely, it should be the composition

$$c: HF(L_1, \tau_L(L_2)) \to HF(L, \tau_L(L_2)) \otimes HF(L_1, L) \cong$$
$$\cong HF(L, L_2) \otimes HF(L_1, L)$$

of the coproduct and the isomorphism $HF(L, \tau_L(L_2)) \cong HF(\tau_L^{-1}(L), L_2) = HF(L, L_2)$. One can show that this is indeed the same map as that obtained by inverting (3); the proof uses invariants of the same kind as those defining h, for two-parameter families of exact Morse fibrations.

Remark 3.4. The assumption $2c_1(M, L) = 0$ is a technical one. It implies that the space of (j, J)-holomorphic sections in Example 3.1 has expected dimension 2n - 1. From this it follows (except for n = 1, which requires a separate treatment) that generically there is a positive-dimensional space of these sections u such that $u(z_0)$ is a specific point in L. That enters into the description of the limiting behaviour of sections of the family in Figure 5 when both moveable points of Σ go towards the fixed one, and through it into the proof that h is a homotopy $ba \simeq 0$. Still, it may be possible to substitute some other argument at this point, and thereby remove the assumption from Theorem 3.3.

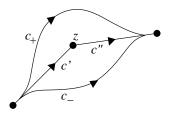
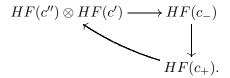


Figure 6.

There is an important special case in which Theorem 3.3 has a graphical interpretation, namely when all Lagrangian submanifolds involved are vanishing cycles. Take an exact Morse fibration (E, π) with arbitrary base S. To any path $c: [0;1] \to S$ with $c^{-1}(S^{\text{crit}}) = \{0;1\}$ and $c'(0), c'(1) \neq 0$ one can associate the Floer cohomology group $HF(V_{c,0}, V_{c,1})$ where $V_{c,0}, V_{c,1} \subset E_{c(t)}$,

for some 0 < t < 1, are the vanishing cycles associated to c|[0;t] resp. c|[t;1]. This group is independent of t, so we may write temporarily HF(c) for it. Consider four paths as in Figure 6. As three of them run together for small t, we may assume that their vanishing cycles all lie in the same fibre, called M. Then $V_{c_{-,0}} = V_{c_{+,0}} = V_{c',0}$ and by the Picard-Lefschetz theorem, $V_{c_{+,1}} = \tau_{V_{c',1}}(V_{c_{-,1}})$. Taking the vanishing cycles of c'' and moving them to M by parallel transport shows that $HF(V_{c',1}, V_{c_{-,1}}) = HF(V_{c',1}, V_{c_{+,1}}) = HF(V_{c'',0}, V_{c'',1})$. Applying Theorem 3.3 in this context yields a long exact sequence



This can be thought of as a skein rule, with $HF(c'') \otimes HF(c')$ measuring the change to $HF(c_{-})$ as the path moves over $z \in S^{\text{crit}}$ and becomes c_{+} . For S = D, allowing oneself for a moment to believe naively that "two terms in a long exact sequence determine the third one", one sees that all groups HF(c) could be determined from knowing just finitely many of them, through a process of successively breaking up paths into shorter pieces.

References. Our TQFT formalism for Floer cohomology is a variation of that in [14], which in turn generalizes earlier work of Piunikhin-Salamon-Schwarz [11]. The pair-of-pants product and continuation map are defined in [10, Chapter 10] or [13] or [17], respectively [12]. The skein rule interpretation of the exact sequence is due to Donaldson.

4. Grading

(4A) The assumption $2c_1(M, L) = 0$ in Theorem 3.3 is very close to the condition needed to equip Floer cohomology groups with \mathbb{Z} -gradings, so one might just as well take advantage of it. Doing that requires some preliminaries, which we now go through.

Let M be an arbitrary symplectic manifold, and \mathcal{J}_M the space of all compatible almost complex structures. There is a canonical unitary line bundle $\Delta_M \to \mathcal{J}_M \times M$ whose fibre at (J,x) is $\Lambda^n(TM_x,J)^{\otimes 2}$. To any Lagrangian submanifold $L \subset M$ one can associate a section $\det^2(TL)$ over $\mathcal{J}_M \times L$ of the associated circle bundle $S(\Delta_M)$; and for any symplectic automorphism ϕ there is a canonical isomorphism $\det^2(D\phi): \Delta_M \to \Delta_M$, covering the map $\phi_* \times \phi$ from $\mathcal{J}_M \times M$ to itself. A Maslov map is a trivialization $\delta_M : \Delta_M \to \mathbb{C}$. Suppose that we have chosen such a δ_M . A grading of $L \subset M$ is a lift

Any two gradings differ by a locally constant function $L \to \mathbb{Z}$; we write $\widetilde{L}[\sigma]$ for $\widetilde{L} - \sigma$, $\sigma \in \mathbb{Z}$. Similarly, a grading of ϕ is a map $\widetilde{\phi} : \mathcal{J}_M \times M \to \mathbb{R}$ such that $\exp(2\pi i \widetilde{\phi}(J,x)) = \delta_M(\det^2(D\phi)(w))/\delta_M(w)$ for any $w \in S(\Delta_M)$ in the fibre over (J,x). When dealing with connected manifolds M with boundary and maps ϕ trivial near ∂M , there is often a preferred grading, characterized by being zero near $\mathcal{J}_M \times \partial M$. Gradings of L and ϕ induce a grading of $\phi(L)$:

$$\tilde{\phi}(\widetilde{L}) \stackrel{\text{def}}{=} (\tilde{\phi} + \widetilde{L}) \circ (\phi_*^{-1} \times \phi^{-1}).$$

If $L \subset M$ is a framed Lagrangian sphere admitting gradings, there is a distinguished grading $\tilde{\tau}_L$ of τ_L , which is zero outside $\mathcal{J}_M \times \{\text{neighbourhood of } L\}$ and satisfies

(4)
$$\widetilde{\tau}_L(\widetilde{L}) = \widetilde{L}[1-n].$$

Graded Lagrangian submanifolds and graded symplectic automorphisms are pairs consisting of an L resp. ϕ together with a choice of grading. For brevity we denote such pairs by $\widetilde{L},\widetilde{\phi}$ only. The Floer cohomology of a pair of graded Lagrangian submanifolds, whenever defined (e.g. when M is exact with contact type boundary, and the Lagrangian submanifolds are themselves exact), has a canonical \mathbb{Z} -grading with the properties

$$HF^{*}(\widetilde{L}_{1}[-\sigma], \widetilde{L}_{2}) = HF^{*}(\widetilde{L}_{1}, \widetilde{L}_{2}[\sigma]) = HF^{*+\sigma}(\widetilde{L}_{1}, \widetilde{L}_{2}),$$

$$(5) \qquad HF^{*}(\widetilde{\phi}(\widetilde{L}_{1}), \widetilde{\phi}(\widetilde{L}_{2})) \cong HF^{*}(\widetilde{L}_{1}, \widetilde{L}_{2}),$$

$$HF^{*}(\widetilde{L}_{2}, \widetilde{L}_{1}) \cong HF^{n-*}(\widetilde{L}_{1}, \widetilde{L}_{2})^{\vee}.$$

- (4B) Now consider an exact Morse fibration (E,π) over S, and the space $\mathcal{J}_{E/S}$ of pairs (j,J) such that J is compatible relative to j. Let $\Delta_{E/S} \to \mathcal{J}_{E/S} \times E$ be the line bundle with fibres $\Lambda^{n+1}(TE_x,J)^{\otimes 2} \otimes (TS_{\pi(x)},j)^{\otimes -2}$. A **relative Maslov map** is a trivialization $\delta_{E/S}:\Delta_{E/S}\to\mathbb{C}$. Once such a map has been chosen, there is a canonical induced Maslov map δ_M on any regular fibre $M=E_z$. The vanishing cycles $V\subset M$ associated to paths $c:[0;1]\to S$, c(0)=z and $c(1)\in S^{\mathrm{crit}}$, admit gradings; and the monodromies ρ_l along closed loops l in $S\setminus S^{\mathrm{crit}}$, l(0)=z, even have canonical gradings, which we denote by $\tilde{\rho}_l$. If $[l]\in H_1(S)$ vanishes, $\tilde{\rho}_l$ is zero near $\mathcal{J}_M\times\partial M$. This implies a graded version of the Picard-Lefschetz theorem, saying that $\tilde{\rho}_l=\tilde{\tau}_V$ for c,l as in Figure 1.
- (4c) Any Lagrangian boundary condition Q for (E, π) comes with a canonical section of $S(\Delta_{E/S})|\mathcal{J}_{E/S} \times Q$. We call $\delta_{E/S}$ adapted to Q if there is a diagram

$$\begin{array}{ccc} \mathcal{J}_{E/S} \times Q & \xrightarrow{\text{canonical section}} & S(\Delta_{E/S}) \\ \downarrow^{\text{id} \times \pi} & & \downarrow^{\delta_{E/S}} \\ \mathcal{J}_{E/S} \times \partial S - - - - - \xrightarrow{\lambda} - - - - \to S^{1}. \end{array}$$

For compact S, the existence of an adapted relative Maslov map means that all components of the space of (j, J)-holomorphic sections taking ∂S to Q have the same expected dimension $n \chi(S) + \deg(\lambda)$; we have already seen an instance of this in Remark 3.4. There is a generalization of this notion to the relative case, which ensures that the Floer groups in (1) have canonical gradings and that the invariant Φ_{rel} is of a specific degree, not necessarily zero. Instead of explaining this in general we will just look at an example, that of the long exact sequence.

Suppose then that we have an exact symplectic manifold M with a Maslov map δ_M , graded Lagrangian submanifolds $\widetilde{L}_1, \widetilde{L}_2$, and a framed exact Lagrangian sphere L which admits gradings. The map $b: HF^*(\widetilde{L}_1, \widetilde{L}_2) \to HF^*(\widetilde{L}_1, \widetilde{\tau}_L(\widetilde{L}_2))$ has degree zero, and the same holds for the pair-of-pants product $a: HF^*(\widetilde{L}, \widetilde{L}_2) \otimes HF^*(\widetilde{L}_1, \widetilde{L}) \to HF^*(\widetilde{L}_1, \widetilde{L}_2)$. The coproduct $HF^*(\widetilde{L}_1, \widetilde{\tau}_L(\widetilde{L}_2)) \to HF^*(\widetilde{L}, \widetilde{\tau}_L(\widetilde{L}_2)) \otimes HF^*(\widetilde{L}_1, \widetilde{L})$ has degree n because of Poincaré duality, but then by (4) and (5) the isomorphism

$$HF^*(\widetilde{L}, \widetilde{\tau}_L(\widetilde{L}_2)) = HF^*(\widetilde{\tau}_L^{-1}(\widetilde{L}), \widetilde{L}_2) = HF^{*+1-n}(\widetilde{L}, \widetilde{L}_2)$$

has degree 1 - n. Hence the third map c raises degrees by one, like in any other cohomological long exact sequence.

References. Floer's discussion of grading in [4] is essentially complete as it stands. Still, graded Lagrangian submanifolds, introduced by Kontsevich [9] somewhat later, allow a better formulation of the results.

5. MUTATION

(5A) As in our discussion of Floer theory, we take the ground field to be $\mathbb{Z}/2$; all categories will be linear over it. Let \mathcal{C} be a triangulated category such that the spaces $\operatorname{Hom}_{\mathcal{C}}^*(X,Y) = \bigoplus_r \operatorname{Hom}_{\mathcal{C}}(X,Y[r])$ are finite-dimensional for any X,Y. An exceptional collection in \mathcal{C} is a finite family of objects (X^1,\ldots,X^m) satisfying

$$\operatorname{Hom}_{\operatorname{\mathcal{C}}}^*(X^i, X^k) = \begin{cases} \mathbb{Z}/2 \cdot \operatorname{id}_{X^i} & i = k, \\ 0 & i > k. \end{cases}$$

Such a collection is called full if the X^i generate \mathcal{C} in the triangulated sense. For $X,Y\in \mathrm{Ob}\,\mathcal{C}$ define $T_X(Y)$, up to isomorphism, as the object fitting into an exact triangle

$$T_X(Y)[-1] \to \operatorname{Hom}_{\mathcal{C}}^*(X,Y) \otimes X \xrightarrow{ev} Y \to T_X(Y);$$

here the tensor product is just a finite sum of shifted copies of X, and ev the canonical evaluation map. If (X^1, \ldots, X^m) is a full exceptional collection then so are $(Y^1, \ldots, Y^m), (Z^1, \ldots, Z^m)$ where

(6)
$$Y^{i} = \begin{cases} T_{X^{1}}(X^{i+1}) & i < m, \\ X^{1} & i = m, \end{cases}$$

respectively

(7)
$$Z^{i} = \begin{cases} X^{i} & i < m - 1, \\ T_{X^{m-1}}(X^{m}) & i = m - 1, \\ X^{m-1} & i = m. \end{cases}$$

(5B) There is a slightly different version of the same story for A_{∞} -categories. Call an A_{∞} -category \mathcal{A} directed if it has finitely many objects numbered $1, \ldots, m$, say $\mathrm{Ob}\,\mathcal{A} = \{X^1, \ldots, X^m\}$, such that

$$hom_{\mathcal{A}}(X^{i}, X^{k}) = \begin{cases} \text{finite-dimensional} & i < k, \\ \mathbb{Z}/2 \cdot \text{id}_{X^{i}} & i = k, \\ 0 & i > k. \end{cases}$$

Note that $\mu_{\mathcal{A}}^d = 0$ is necessarily zero for $d > \max\{m-1,2\}$. We recall the definition of the (bounded) derived category $D^b(\mathcal{A})$. The first step is to embed \mathcal{A} into a larger A_{∞} -category \mathcal{A}^{\oplus} which has finite sums and shifts. Thus, an object of \mathcal{A}^{\oplus} is a formal sum $\bigoplus_{e \in E} X_e[\sigma_e]$ with E a finite set, $X_e \in \operatorname{Ob} \mathcal{A}$, and $\sigma_e \in \mathbb{Z}$. Next, a twisted complex in \mathcal{A} is a pair (C, δ_C) consisting of $C \in \operatorname{Ob} \mathcal{A}^{\oplus}$ and $\delta_C \in \operatorname{hom}_{\mathcal{A}^{\oplus}}^1(C, C)$, such that the "generalized Maurer-Cartan equation"

(8)
$$\sum_{d>1} \mu_{\mathcal{A}^{\oplus}}^{d}(\delta_{C}, \dots, \delta_{C}) = 0$$

holds. Twisted complexes form an A_{∞} -category $\operatorname{Tw} \mathcal{A}$ which again has direct sums and shifts. It also contains a cone $\operatorname{Cone}(a)$ for any morphism a such that $\mu^1_{\operatorname{Tw} \mathcal{A}}(a) = 0$, and the cohomological category $D^b(\mathcal{A}) = H^0(\operatorname{Tw} \mathcal{A})$ inherits a triangulated structure from this. The objects of \mathcal{A} , seen as twisted complexes with zero differential, form a full exceptional collection in $D^b(\mathcal{A})$.

Remark 5.1. The definition of the derived category of a general A_{∞} -category A uses pairs (C, δ_C) such that δ_C is strictly decreasing with respect to some finite filtration of C, which has the effect of making the sum (8) finite. The fact that this is not necessary in our case is just one of several technical simplifications which directedness brings with it.

Any A_{∞} -functor $F: \mathcal{A} \to \mathcal{B}$ between directed A_{∞} -categories induces another one $\operatorname{Tw} F: \operatorname{Tw} \mathcal{A} \to \operatorname{Tw} \mathcal{B}$ taking cones to cones, and therefore an exact functor $D^b(F) = H^0(\operatorname{Tw} F): D^b(\mathcal{A}) \to D^b(\mathcal{B})$. Call F a quasi-isomorphism if $H(F): H(\mathcal{A}) \to H(\mathcal{B})$ is an isomorphism; since the objects of \mathcal{A} generate $D^b(\mathcal{A})$, it follows that in this case $D^b(F)$ is an equivalence.

More interestingly, suppose that \mathcal{A} is some directed A_{∞} -category, and (Y^1, \ldots, Y^m) an exceptional collection in $D^b(\mathcal{A})$. One can then define a new directed A_{∞} -category \mathcal{B} , called the directed A_{∞} -subcategory of $\operatorname{Tw} \mathcal{A}$ generated by the Y^i , as follows. Objects of \mathcal{B} are the Y^i , and $hom_{\mathcal{B}}(Y^i, Y^k) = hom_{\operatorname{Tw} \mathcal{A}}(Y^i, Y^k)$ for i < k, with the same composition maps between these groups as in $\operatorname{Tw} \mathcal{A}$. All other morphism groups and compositions are as

dictated by the directedness condition. The embedding $\mathcal{B} \hookrightarrow \operatorname{Tw} \mathcal{A}$ can be extended to an A_{∞} -functor $\iota : \operatorname{Tw} \mathcal{B} \to \operatorname{Tw} \mathcal{A}$, which induces an exact functor $H^0(\iota) : D^b(\mathcal{B}) \to D^b(\mathcal{A})$. For the same reason as before, $H^0(\iota)$ is full and faithful; it is even an equivalence if (Y^1, \ldots, Y^m) is a full exceptional collection.

(5c) Given $X \in \text{Ob Tw } \mathcal{A}$ and a finite-dimensional complex V of vector spaces, one can form the tensor product $V \otimes X \in \text{Ob Tw } \mathcal{A}$, which is a direct sum of shifted copies of X with a differential combining those on V and X. Taking $V = (hom_{\mathrm{Tw } \mathcal{A}}(X,Y), \mu_{\mathcal{A}}^1)$ for some $Y \in \text{Ob Tw } \mathcal{A}$, one has a canonical evaluation morphism $ev \in hom_{\mathrm{Tw } \mathcal{A}}^0(hom_{\mathrm{Tw } \mathcal{A}}(X,Y) \otimes X,Y)$ with $\mu_{\mathrm{Tw } \mathcal{A}}^1(ev) = 0$. Let $T_X(Y) \in \text{Ob Tw } \mathcal{A}$ be the cone of ev. This is isomorphic in $D^b(\mathcal{A})$ to the object of the same name introduced above, but it is now unique in a strict sense, not just up to isomorphism. Taking the exceptional collection formed by the objects of \mathcal{A} and applying (6) or (7) yields another full exceptional collection, hence a directed A_{∞} -subcategory \mathcal{B} of $\mathrm{Tw } \mathcal{A}$ with $D^b(\mathcal{B}) \cong D^b(\mathcal{A})$. This process can be repeated indefinitely and leads to the following notion:

Definition 5.2. Two directed A_{∞} -categories with m objects are mutations of each other if they can be related by a sequence of the following moves and their inverses:

- $A \leadsto B$ if there is a quasi-isomorphism between them.
- It is allowed to shift each object by some degree, which means changing the grading of each group $hom_{\mathcal{A}}(X^i, X^k)$ by $(\sigma_i \sigma_k)$ for some $\sigma_1, \ldots, \sigma_m \in \mathbb{Z}$, while keeping the same composition maps.
- $A \rightsquigarrow cA$ where, if the objects of cA are denoted by $\{Y^1, \ldots, Y^m\}$, the nontrivial morphism spaces are

$$hom_{cA}(Y^{i}, Y^{k}) = \begin{cases} hom_{A}(X^{i+1}, X^{k+1}) & i < k < m, \\ hom_{A}(X^{1}, X^{i+1})^{\vee}[-1] & i < m, k = m. \end{cases}$$

Here $^{\vee}$ denotes the dual of a graded vector space. The compositions $\mu_{cA}^d:\prod_{\nu=1}^d hom_{cA}(Y^{i_{\nu}},Y^{i_{\nu+1}}) \to hom_{cA}(Y^{i_1},Y^{i_{d+1}}), \ i_1<\dots< i_{d+1},$ are equal to those in A except when $i_{d+1}=m$, in which case one has $\langle \mu_{cA}^d(a^d,\dots,a^1),b\rangle=\langle a^d,\mu_A^d(a^{d-1},\dots,a^1,b)\rangle$ with $\langle\dots\rangle$ the dual pairing.

• $\mathcal{A} \leadsto r\mathcal{A}$ with $\operatorname{Ob} r\mathcal{A} = \{Z^1, \dots, Z^m\}$ and the following nontrivial morphism spaces $\operatorname{hom}_{r\mathcal{A}}(Z^i, Z^k)$:

(9)
$$\begin{cases} hom_{\mathcal{A}}(X^{i}, X^{k}) & i < k \leq m-2, \\ hom_{\mathcal{A}}(X^{i}, X^{m-1}) & i \leq m-2, k = m, \\ hom_{\mathcal{A}}(X^{m-1}, X^{m})^{\vee}[-1] & i = m-1, k = m, \\ (hom_{\mathcal{A}}(X^{m-1}, X^{m}) \otimes hom_{\mathcal{A}}(X^{i}, X^{m-1}))[1] \oplus \\ \oplus hom_{\mathcal{A}}(X^{i}, X^{m}) & i \leq m-2, k = m-1. \end{cases}$$

 μ_{rA}^1 is given by μ_A^1 in the first two cases, its dual $(\mu_A^1)^{\vee}$ in the third, and in the final case by

$$\begin{pmatrix} \mu_{\mathcal{A}}^1 \otimes \mathrm{id} + \mathrm{id} \otimes \mu_{\mathcal{A}}^1 & 0 \\ \mu_{\mathcal{A}}^2 & \mu_{\mathcal{A}}^1 \end{pmatrix}.$$

Most of the higher order maps

(10)
$$\mu_{r,A}^d: \prod_{\nu=1}^d hom_{r,A}(Z^{i_{\nu}}, Z^{i_{\nu+1}}) \to hom_{r,A}(Z^{i_1}, Z^{i_{d+1}})$$

for $i_1 < \cdots < i_{d+1}$ are taken from those of A in a straightforward way, but there are two exceptions. One is when $i_{d+1} = m-1$, in which case

$$\mu_{r\mathcal{A}}^d = \begin{pmatrix} \mathrm{id} \otimes \mu_{\mathcal{A}}^d & 0\\ \mu_{\mathcal{A}}^{d+1} & \mu_{\mathcal{A}}^d \end{pmatrix}$$

with respect to the obvious splittings on both sides of (10). The second exceptional case is when $i_d = m - 1$, $i_{d+1} = m$: then μ_{rA}^2 is

$$hom_{\mathcal{A}}(X^{m-1}, X^{m})^{\vee} \otimes hom_{\mathcal{A}}(X^{m-1}, X^{m}) \otimes hom_{\mathcal{A}}(X^{i_{1}}, X^{m-1})$$

$$\oplus (hom_{\mathcal{A}}(X^{m-1}, X^{m})^{\vee} \otimes hom_{\mathcal{A}}(X^{i_{1}}, X^{m}))[-1]$$

$$\downarrow hom_{\mathcal{A}}(X^{i_{1}}, X^{m-1}),$$

$$\mu_{r,d}^{2}(a^{3} \otimes a^{2} \otimes a^{1}, b^{2} \otimes b^{1}) = \langle a^{3}, a^{2} \rangle a^{1},$$

while the compositions of order $d \geq 3$ vanish.

Actually, while $r\mathcal{A}$ is precisely the directed A_{∞} -subcategory of Tw \mathcal{A} generated by the collection (7), $c\mathcal{A}$ is only canonically quasi-isomorphic to that generated by (6). But it is still true that two directed A_{∞} -categories which are mutations of each other have equivalent derived categories.

References. This section makes no claim to originality. Exceptional collections in triangulated categories are discussed in the work of Bondal, Gorodentsev, Kapranov, and others. Most relevant for us is [2] which emphasizes the role of dg categories. The short step from there to A_{∞} -categories was made by Kontsevich [8], who is also responsible for introducing Tw \mathcal{A} and $D^b(\mathcal{A})$ [9].

6. Fukaya categories

(6A) Throughout this section M is a fixed exact symplectic manifold, with a Maslov map δ_M . A graded Lagrangian configuration in M is a family $\widetilde{\Gamma} = (\widetilde{L}_1, \dots, \widetilde{L}_m)$ of graded, exact, framed Lagrangian spheres. Hurwitz equivalence for graded configurations is defined as in the ungraded case, with the following adaptations: for the isotopy invariance one wants to take a grading $\widetilde{\phi}$ which is zero near $\mathcal{J}_M \times \partial M$, so that $\widetilde{\phi}$ is isotopic to the identity in the group of graded symplectic automorphisms; the moves $c\widetilde{\Gamma}, r\widetilde{\Gamma}$ use the canonical gradings $\widetilde{\tau}_L$ of Dehn twists; and there is an additional shift move,

•
$$\widetilde{\Gamma} \leadsto (\widetilde{L}_1[\sigma_1], \ldots, \widetilde{L}_m[\sigma_m])$$
 for any $\sigma_1, \ldots, \sigma_m \in \mathbb{Z}$.

This is something of an anticlimax, since it cancels out the extra information contained in the grading; but in fact, the whole notion of graded configuration has been introduced only for notational convenience.

We will associate to Γ a directed Fukaya A_{∞} -category $Lag^{\rightarrow}(\Gamma)$, unique up to quasi-isomorphism. Suppose first that the configuration is in general position, meaning that any two L_i are transverse and there are no triple intersections. Objects of $\mathcal{A} = Lag^{\rightarrow}(\Gamma)$ are the graded Lagrangian submanifolds L_i , in the given order, and

$$hom_{\mathcal{A}}(\widetilde{L}_{i}, \widetilde{L}_{k}) = \begin{cases} CF^{*}(\widetilde{L}_{i}, \widetilde{L}_{k}) = (\mathbb{Z}/2)^{L_{i} \cap L_{k}} & i < k, \\ \mathbb{Z}/2 \cdot \operatorname{id}_{\widetilde{L}_{i}} & i = k, \\ 0 & i > k. \end{cases}$$

Roughly speaking, $\mu_{\mathcal{A}}^1$ is the Floer boundary map, $\mu_{\mathcal{A}}^2$ the pair-of-pants product, and $\mu_{\mathcal{A}}^3, \mu_{\mathcal{A}}^4, \ldots$ Fukaya's generalizations of that product. Each $\mu_{\mathcal{A}}^d$ depends on the choice of a family \mathbf{J}^{d+1} of almost complex structures on M, and these choices have to obey certain consistency conditions, which we will now outline (this is joint work of Lazzarini and the author).

In a first step one takes, for each $1 \le i_1 < i_2 \le m$, a generic family of almost complex structures $\mathbf{J}^2(i_1,i_2):[0;1]\to\mathcal{J}_M$. As is well known, this causes all solutions of Floer's equations

(11)
$$u: \mathbb{R} \times [0;1] \to M, \qquad \begin{cases} \mathbf{J}^2(i_1, i_2, t) \circ du(s, t) = du(s, t) \circ j, \\ u(\mathbb{R} \times \{1\}) \subset L_{i_1}, \ u(\mathbb{R} \times \{0\}) \subset L_{i_2}, \\ \int u^* \omega < \infty, \end{cases}$$

where j is the standard complex structure on $\mathbb{R} \times [0;1]$, to be regular. From the one-dimensional solution spaces (zero-dimensional after dividing

by translation) one builds $\mu_{\mathcal{A}}^1: CF^*(\widetilde{L}_{i_1}, \widetilde{L}_{i_2}) \to CF^{*+1}(\widetilde{L}_{i_1}, \widetilde{L}_{i_2})$. Next take S=D, three cyclically ordered marked points $\zeta_{\nu}^3 \in \partial S$, $1 \leq \nu \leq 3$, local coordinates $\psi_{\nu}^3: D^+ \to S$ around them, and set $S^* = S \setminus \{\zeta_1^3, \zeta_2^3, \zeta_3^3\}$, all as in the definition of the invariants Φ_{rel} . Choose for each $1 \leq i_1 < i_2 < i_2 < i_3 < i_4 < i_4 < i_5 < i_5 < i_6 < i_6 < i_6 < i_7 < i_8 < i_$ $i_3 \leq m$ a generic family $\mathbf{J}^3(i_1,i_2,i_3): S^* \to \mathcal{J}_M$, such that for $s \ll 0$ and $t \in [0; 1],$

(12)
$$\mathbf{J}^{3}(i_{1}, i_{2}, i_{3}, \psi_{\nu}^{3}(e^{\pi(s+it)})) = \begin{cases} \mathbf{J}^{2}(i_{\nu}, i_{\nu+1}, t) & \nu = 1, 2, \\ \mathbf{J}^{2}(i_{1}, i_{3}, 1 - t) & \nu = 3. \end{cases}$$

Denote by
$$I_{\nu}^{3}$$
, $1 \leq \nu \leq 3$, the connected components of ∂S^{*} , ordered cyclically such that I_{1}^{3} lies between ζ_{3}^{3} and ζ_{1}^{3} . The equation defining $\mu_{\mathcal{A}}^{2}$ is
$$u: S^{*} \to M, \qquad \begin{cases} \mathbf{J}^{3}(i_{1}, i_{2}, i_{3}, z) \circ du(z) = du(z) \circ j, \\ u(I_{\nu}^{3}) \subset L_{i_{\nu}} \quad \text{for } \nu = 1, 2, 3, \\ \int u^{*}\omega < \infty. \end{cases}$$

Condition (12) causes this to agree with a suitable equation (11) in the tubular coordinates $\psi_{\nu}^{3}(e^{\pi(s+it)})$ on each end of S^{*} .

The additional ingredient in the definition of the products of order d > 2 are "moduli parameters" as the complex structure of the domain changes. Let $\mathcal{C}^{d+1} \subset (\partial D)^{d+1}$ be the configuration space of d+1 distinct, numbered and cyclically ordered points on ∂D . The moduli space \mathcal{R}^{d+1} and the universal disc bundle \mathcal{S}^{d+1} over it are defined as

$$\mathbb{S}^{d+1} = \mathbb{C}^{d+1} \times_{\operatorname{Aut}(D)} D \longrightarrow \mathbb{R}^{d+1} = \mathbb{C}^{d+1} / \operatorname{Aut}(D),$$

where $\operatorname{Aut}(D) \cong PSL(2,\mathbb{R})$ is the holomorphic automorphism group. Each fibre \mathcal{S}^{d+1}_r carries a canonical complex structure, and there are canonical sections $\zeta^{d+1}_{\nu}: \mathbb{R}^{d+1} \to \partial \mathbb{S}^{d+1}$, $1 \leq \nu \leq d+1$, such that the points $\zeta^{d+1}_{\nu}(r) \in \partial \mathbb{S}^{d+1}_r$ are distinct and cyclically ordered for each r. Write

$$S^{d+1,*} = S^{d+1} \setminus (\bigcup_{\nu} \operatorname{im} \zeta_{\nu}^{d+1}), \qquad S_r^{d+1,*} = S^{d+1,*} \cap S_r^{d+1}.$$

For each $1 \leq i_1 < \dots i_{d+1} \leq m$ one has to choose a family $\mathbf{J}^{d+1}(i_1, \dots, i_{d+1})$: $\mathbb{S}^{d+1,*} \to \mathcal{J}_M$, subject to two kinds of conditions.

- Compatibility with \mathbf{J}^2 . This requires a preliminary choice of maps $\psi_{\nu}^{d+1}: \mathcal{R}^{d+1} \times D^+ \to \mathbb{S}^{d+1}, \ 1 \leq \nu \leq d+1$, such that $\psi_{\nu}^{d+1}(r,\cdot)$ provides local coordinates around $\zeta_{\nu}^{d+1}(r)$ for each $r \in \mathcal{R}^{d+1}$. Then the conditions are similar to (12), requiring $\mathbf{J}^{d+1}(i_1,\ldots,i_{d+1},\psi_{\nu}^{d+1}(r,z))$ to be determined by the previously chosen \mathbf{J}^2 for small |z|.
- Compatibility with \mathbf{J}^{e+1} for $2 \leq e < d$. For this it is necessary to consider the Deligne-Mumford compactification of \mathbb{R}^{d+1} . Each stratum at infinity is a product of lower order spaces \mathbb{R}^{e+1} , and for a point $r \in \mathbb{R}^{d+1}$ sufficiently close to one such stratum, the fibre $\mathbb{S}_r^{d+1,*}$ is built by gluing together fibres of $\mathbb{S}^{e+1,*}$ for the various occurring e. The precise condition on $\mathbf{J}^{d+1}(i_1,\ldots,i_{d+1})|\mathbb{S}_r^{d+1,*}$ is too complicated to be written down here, but informally it says that this should be built up from the \mathbf{J}^{e+1} in a corresponding way.

Let $I_{r,\nu}^{d+1}$, $1 \leq \nu \leq d+1$, be the connected components of $\partial \mathcal{S}_r^{d+1,*}$, ordered cyclically so that $I_{r,1}^{d+1}$ lies between $\zeta_{d+1}^{d+1}(r)$ and $\zeta_1^{d+1}(r)$. The consistency conditions leave enough freedom to make solutions of the equation

$$r \in \mathbb{R}^{d+1}, \ u : \mathbb{S}_r^{d+1,*} \to M, \quad \begin{cases} \mathbf{J}^{d+1}(i_1, \dots, i_{d+1}, z) \circ du(z) = du(z) \circ j, \\ u(I_{r,\nu}^{d+1}) \subset L_{i_{\nu}} \quad \text{for } \nu = 1, \dots, d+1, \\ \int u^* \omega < \infty. \end{cases}$$

regular for generic $\mathbf{J}^{d+1}(i_1,\ldots,i_{d+1})$. Counting such solutions defines

$$\mu_{\mathcal{A}}^d: CF^*(\widetilde{L}_{i_d}, \widetilde{L}_{i_{d+1}}) \otimes \cdots \otimes CF^*(\widetilde{L}_{i_1}, \widetilde{L}_{i_2}) \longrightarrow CF^{*+2-d}(\widetilde{L}_{i_1}, \widetilde{L}_{i_{d+1}}).$$

The dependence of $Lag^{\rightarrow}(\widetilde{\Gamma})$ on the choice of almost complex structure can be analyzed using a one-parameter family argument. Since only finitely many moduli spaces are involved, the A_{∞} -structure is subject to a finite number of changes in the family. At each of these exceptional times one can produce a quasi-isomorphism relating the old A_{∞} -structure with the new one.

Remark 6.1. As a technical point, note that directedness allows us to bypass some problems which plague Fukaya's original setup, having to do with the chain complexes underlying HF(L,L) and unit elements in them.

The next step is isotopy invariance which, as always in Floer theory, is also used to extend the definition to configurations which are not in general position.

Proposition 6.2. Let $\widetilde{\Gamma} = (\widetilde{L}_1, \dots, \widetilde{L}_m)$ be a graded Lagrangian configuration in general position. Take $l \in \{1, \dots, m\}$, a symplectic automorphism ϕ isotopic to the identity, and a grading $\widetilde{\phi}$ which is zero near $\mathcal{J}_M \times \partial M$, such that $\widetilde{\Xi} = (\widetilde{L}_1, \dots, \widetilde{L}_{l-1}, \widetilde{\phi}(\widetilde{L}_l), \dots, \widetilde{\phi}(\widetilde{L}_m))$ is again in general position. Then there is a quasi-isomorphism $F : Lag^{\rightarrow}(\widetilde{\Gamma}) \to Lag^{\rightarrow}(\widetilde{\Xi})$.

We will spend a moment discussing the structure of the proof, since it is a good example of arguments involving directed Fukaya categories. Recall that an A_{∞} -functor $F: \mathcal{A} \to \mathcal{B}$ consists of a map $F: \mathrm{Ob}\,\mathcal{A} \to \mathrm{Ob}\,\mathcal{B}$, chain maps $F^1: hom_{\mathcal{A}}(X,Y) \to hom_{\mathcal{B}}(FX,FY)$ for $X,Y \in \mathrm{Ob}\,\mathcal{A}$, and multilinear "higher order terms" $F^d, d \geq 2$. In the present case, the map on objects is the obvious one, and the nontrivial chain maps

$$F^1: CF^*(\widetilde{L}_i, \widetilde{L}_k) \longrightarrow CF^*(\widetilde{L}_i, \widetilde{\phi}(\widetilde{L}_k)), \quad i < l \text{ and } k \ge l,$$

are those underlying the continuation homomorphisms, so that F is automatically a quasi-isomorphism. The main effort goes into defining higher order terms which satisfy the equations for an A_{∞} -functor.

(6B) We will now describe the relation between Hurwitz moves of $\widetilde{\Gamma}$ and mutations of $D^bLag^{\rightarrow}(\widetilde{\Gamma})$. Proposition 6.2 says that isotopies of $\widetilde{\Gamma}$ result in a quasi-isomorphism. Shifting the gradings \widetilde{L}_i obviously corresponds to the first mutation in Definition 5.2.

Lemma 6.3. $Lag^{\rightarrow}(c\widetilde{\Gamma})$ is quasi-isomorphic to $cLag^{\rightarrow}(\widetilde{\Gamma})$.

The proof relies on the $\mathbb{Z}/(d+1)$ -action on \mathbb{R}^{d+1} given by a cyclic shuffle of the marked points (this symmetry had not been used in the definition of directed Fukaya categories).

Theorem 6.4. $rLag^{\rightarrow}(\widetilde{\Gamma})$ is quasi-isomorphic to $Lag^{\rightarrow}(r\widetilde{\Gamma})$.

To see why this is plausible, set $\mathcal{A} = Lag^{\rightarrow}(\widetilde{\Gamma})$, and let (Z^1, \ldots, Z^m) be the objects of $r\mathcal{A}$. The cohomology $H(hom_{r\mathcal{A}}(Z^i, Z^k), \mu^1_{r\mathcal{A}}), i < k$, is

$$\begin{cases} HF^*(\widetilde{L}_i, \widetilde{L}_k) & i < k \le m-2, \\ HF^*(\widetilde{L}_i, \widetilde{L}_{m-1}) & i \le m-2, k = m, \\ HF^*(\widetilde{\tau}_{L_{m-1}}(\widetilde{L}_m), \widetilde{L}_{m-1}) & i = m-1, k = m, \\ H(\operatorname{Cone}(\mu_{\mathcal{A}}^2 : CF^*(\widetilde{L}_{m-1}, \widetilde{L}_m) \otimes CF^*(\widetilde{L}_i, \widetilde{L}_{m-1}) \to \\ \to CF^*(\widetilde{L}_i, \widetilde{L}_m))) & i \le m-2, k = m-1. \end{cases}$$

This is just (9) except that in writing down the third case we have used Poincaré duality and (4). As we saw when discussing Theorem 3.3, the cone in the last line is isomorphic to $HF^*(\widetilde{L}_i, \widetilde{\tau}_{L_{m-1}}(\widetilde{L}_m))$. Therefore all cohomology groups are in fact isomorphic to the corresponding ones in $Lag^{\rightarrow}(r\widetilde{\Gamma})$. The remainder of the proof, as in Proposition 6.2, consists in extending this to a full-fledged A_{∞} -functor.

By the general theory of mutations, what we have shown implies that if two graded Lagrangian configurations in M are Hurwitz equivalent, their directed Fukaya categories have equivalent derived categories. Combining this with Picard-Lefschetz theory yields the following consequence:

Corollary 6.5. Let (E, π) be an exact Morse fibration over D, with a relative Maslov map $\delta_{E/D}$. Make an admissible choice of paths, and let Γ be the corresponding distinguished basis of vanishing cycles in a fibre. Choose any gradings $\widetilde{\Gamma}$ and form $D^bLag^{\rightarrow}(\widetilde{\Gamma})$. This is independent of all choices up to equivalence, and hence is an invariant of (E, π) and $\delta_{E/D}$.

(6c) There is a computational aspect which Corollary 6.5 fails to convey, and which we will explain by giving an example. Let M be an exact symplectic four-manifold with $2c_1(M)=0$, and (L_1,L_2) two Lagrangian spheres in M. $L'_2=\tau^2_{L_1}(L_2)$ and L_2 are always isotopic as smooth submanifolds. There are two cases, $L_1=L_2$ and $L_1\cap L_2=\emptyset$, in which L'_2 is also Lagrangian isotopic to L_2 for obvious reasons, but in general this is false:

Proposition 6.6. Suppose that L_1, L_2 intersect transversally, with $|L_1 \cap L_2| \geq 3$, and that the local intersection numbers at all points are the same. Then L'_2 is not Lagrangian isotopic to L_2 .

The proof goes as follows. Choose a Maslov map δ_M and gradings $\widetilde{L}_1, \widetilde{L}_2$. The directed Fukaya category $\mathcal{A} = Lag^{\rightarrow}(\widetilde{\Gamma})$ of the configuration $\widetilde{\Gamma} = (\widetilde{L}_1, \widetilde{L}_1, \widetilde{L}_2, \widetilde{L}_2)$ is determined up to quasi-isomorphism by the graded vector space $R = HF^*(\widetilde{L}_1, \widetilde{L}_2)$ together with the degree two maps $q_1, q_2 \in \operatorname{End}(R)$ given by the pair-of-pants product with the unique nontrivial element in $HF^2(\widetilde{L}_1, \widetilde{L}_1)$ resp. $HF^2(\widetilde{L}_2, \widetilde{L}_2)$. These satisfy $q_1 \circ q_2 = q_2 \circ q_1$ and $q_1^2 = q_2^2 = 0$. Lemma 6.3 and Theorem 6.4 give an explicit sequence of mutations which transforms \mathcal{A} into the directed Fukaya category \mathcal{B} associated to the Hurwitz equivalent configuration

$$ccrc^{-1}rc^{-1}\widetilde{\Gamma} = (\widetilde{\tau}_{L_1}^2(\widetilde{L}_2), \widetilde{L}_1, \widetilde{L}_1, \widetilde{L}_2).$$

In particular this determines $HF^*(\tilde{\tau}_{L_1}^2(\tilde{L}_2), \tilde{L}_2)$, since that is a morphism space in $H(\mathcal{B})$. We will not write down the actual computation; the outcome is the total cohomology of the complex

$$\mathbb{Z}/2 \oplus \mathbb{Z}/2[-2] \xrightarrow{(\mathrm{id},q_2)} \mathrm{End}(R) \xrightarrow{\psi} \mathrm{End}(R)[2],$$

where the second arrow is $\psi(x) = q_1 \circ x - x \circ q_1$. Since $\psi^2(x) = 2q_1 \circ x \circ q_1 = 0$, linear algebra tells us that the dimension of $\operatorname{coker}(\psi)$ is $\geq (\dim R)^2/2$.

With the assumption dim $R \geq 3$ this implies that $HF(L'_2, L_2)$ is bigger than $HF(L_2, L_2)$, which completes the argument.

References. The presence of A_{∞} -structures in Floer theory was discovered by Fukaya [5, 6, 7]; at the time of writing, there are still no published proofs of the basic analytic results. Our approach to transversality is joint work with Lazzarini. The formal resemblance between Hurwitz moves and mutations was pointed out by Kontsevich. With the benefit of hindsight one can see that Theorem 6.4 is, in conjectural form, implicit in his discussion of that phenomenon [8]. Proposition 6.6 takes up a topic discussed in [15] and [16]. However, the result itself is new, and can apparently not be proved by the elementary methods used in those earlier papers.

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